Azimuthal Anisotropy Adaptive Mapping Functions

Pascal Gegout, Richard Biancale, Laurent Soudarin

Groupe de Recherche de Géodésie Spatiale - Toulouse
1 CNRS/GS, 2 CNES/GS, 3 CNES/CLS Analysis Center

Session 1: Technical and Physical Details of Ray-Tracing
Session 2: Models derived from Ray-Traced Delays

Workshop on Ray-Tracing for Space Geodetic Techniques
29-30 April 2010 - Institute of Geodesy and Geophysics - Vienna
1. Refractivity of the Troposphere

- Vertical Hybrid Coordinates
- Equations of State
- Orography and Geopotential
- Model Levels Heights
- Refractivity
- Refractivity Functions
The Integrated Forecasting System (IFS) implemented at ECMWF, the European Center for Medium-range Weather Forecasts, contains a four-dimensional variational (4DVAR) analysis system.

- The 4DVAR analysis is closely equivalent to a Kalman filter analysis.
- The influence of an observation in space and time is controlled by the model dynamics which increases its realism of the spreading out of the information.
- The atmospheric analysis procedure is a four-dimensional variational data assimilation over a time window from 9 hours before to 3 hours after the nominal analysis time (00, 06, 12 and 18 UTC).
- Delayed cutoff data assimilation steps are provided 3-hourly over 12 hours windows at 09 and 21 UTC.
Horizontal Discretisation

- The horizontal representation of the atmospheric model is defined by the spherical harmonics truncation.
  - at a T511L60 resolution from January 2001 to January 2006,
  - at a T799L91 resolution from February 2006 to January 2010,
  - and has increased to a T1279L91 resolution in January 2010.

- A truncation at degree T799 (N400 gaussian grid) corresponds to a 25 km resolution which may be interpolated at a 0.225 by 0.225 regular latitude-longitude grid.

- A truncation at degree T1279 (N640 gaussian grid) corresponds to a 16 km resolution which may be interpolated at a 0.125 by 0.125 regular latitude-longitude grid.
Vertical Discretisation

- The vertical hybrid coordinates are defined by the pressures at half-levels which are functions of the surface pressure $p_s(\phi, \lambda; t)$ at a specific location at a given time.

$$p_{k+1/2}(\phi, \lambda; t) = A_{k+1/2} + B_{k+1/2} p_s(\phi, \lambda; t)$$  \hspace{1cm} (1)

The set of constants $A_{k+1/2}$ and $B_{k+1/2}$ define the vertical hybrid coordinates system varying in time with surface pressure $p_s(\phi, \lambda; t)$.

- The prognostic variables $\Phi_k, T_k, q_k$ are represented by their values at the full-level pressures $p_k$ coherently described within the model’s vertical finite-difference scheme by

$$\ln p_k = \frac{p_{k+1/2} \ln p_{k+1/2} - p_{k-1/2} \ln p_{k-1/2}}{\Delta p_k} - 1$$ \hspace{1cm} (2)

$$\Delta p_k = p_{k+1/2} - p_{k-1/2}.$$ \hspace{1cm} (3)
The atmosphere is made of moist air, a mix of dry air and water vapor. The equation of state for dry air and water vapor are given by

\[ p_{\text{dry}} = \rho_{\text{dry}} R_{\text{dry}} T \quad p_{\text{vap}} = \rho_{\text{vap}} R_{\text{vap}} T \] (4)

Specific humidity \( q \) is the mass of water vapor per unit mass of dry air.

\[ q = \frac{\rho_{\text{vap}}}{\rho_{\text{dry}}} \] (5)

The virtual temperature is the temperature that dry air need to have the same density as moist air at a given pressure.

\[ T_v = T \left(1 + q \left(\frac{R_{\text{vap}}}{R_{\text{dry}}} - 1\right)\right) \] (6)

The pressure \( p \), the density \( \rho \) and the equation of state for moist air are:

\[ p = p_{\text{dry}} + p_{\text{vap}} \quad \rho = \rho_{\text{dry}} + \rho_{\text{vap}} \quad p = \rho R_{\text{dry}} T_v \] (7)
The complete atmospheric structure is recovered from the archived model levels fields: the surface pressure $p_s$, the surface geopotential $\Phi_s$, the temperature $T_k$ and the specific humidity $q_k$.

For a given tracking site only the atmosphere above the local horizon is needed since radio waves do not travel through the solid Earth.

Area Interpolation:
Using the EMOSLIB library provided by ECMWF, the model level fields are read, decoded and interpolated on regular $0.125 \times 0.125$ latitude-longitude subgrids centered on the site’s nearest point with a 12.5 degrees extent (100 samples) from the tracking site.
Figure: Area Orography of the Tsukuba VLBI 7345 site 08/12/2008, 0Z.
Model Levels Heights as defined in Meteorology

- The surface pressure $p_s$ and the surface geopotential $\Phi_s$ are defined at the (surface) orography. The orography is derived by averaging (and small-scales filtering) the GTOPO30 terrain elevation data set.

- The resulting mean topography is realistic over most land areas but is insufficient in high mountain areas where the standard deviation of the topography is added to properly trigger gravity waves.

- The orthometric height of the orography at the geodetic latitude $\phi$ and longitude $\lambda$ is defined implicitly using the conventions of the GTOPO30 elevation model. The elevations of GTOPO30 are referenced vertically to the EGM96 geoid $N(\phi, \lambda)$ of WGS84.

- The conclusion reached is that the surface geopotential $\Phi_s(\phi, \lambda)$ is defined in meteorology as the orthometric height of the orography multiplied by the gravity constant of $g = 9.80665 \text{ ms}^{-2}$ (WMO).
The vertical discretization of the model rely on the surface geopotential and the discrete formulation of the hydrostatic equation for the moist air:

\[
\Phi_{k+1/2} - \Phi_{k-1/2} = -R_{dry} (T_v)_k \ln \left( \frac{p_{k+1/2}}{p_{k-1/2}} \right)
\]  

(8)

and provide the definition of the half-level geopotential using the surface geopotential as reference:

\[
\Phi_{k+1/2} = \Phi_s + \sum_{j=k+1}^{NLEV} R_{dry} (T_v)_j \ln \left( \frac{p_{j+1/2}}{p_{j-1/2}} \right)
\]  

(9)

Full-level values of the geopotential are given by

\[
\Phi_k = \Phi_{k+1/2} + \alpha_k R_{dry} (T_v)_k
\]  

(10)

with \( \alpha_k = 1 - \frac{p_{k-1/2}}{\Delta p_k} \ln \frac{p_{k+1/2}}{p_{k-1/2}} \) \( \forall k > 1 \) and \( \alpha_1 = \ln 2 \)  

(11)
One should notice that in numerical weather modeling no assumption is made on the geometry of the Atmosphere, except the invariant boundary condition provided by the surface geopotential.

The pressure levels are partly homothetically defined from the time variable surface pressure on the orography and geopotential levels are defined from these vertical hybrid coordinates at each time step.

The parametrisation is only made off physical and dynamical parameters defined in the frame of the vertical hybrid coordinates and the influence of gravity on the atmospheric dynamic derives from the geopotential levels.

\[\text{We have to rebuild the geometry of the atmosphere from the dynamical geopotential levels at each time step.}\]

\[\text{In order to build precisely the geometric structure of the atmosphere using a realistic geodetic model, we need to take into account the ellipsoidal shape of the Earth, the shape of the geoid and the gravity acceleration above the ellipsoid.}\]
Model Levels Heights designed for Geodesy

geodetic height \( h = \) orthometric height \( H \) + geoidal undulation \( N \)

height relative to the ellipsoid

height relative to the geoid

geodetic height \( h = \) normal height \( H^* \) + height anomaly \( \zeta \)

A geopotential difference is related to the orthometric height or to the normal height by

\[
\Phi_0 - \Phi_k = \int_0^H g \, dH = \int_0^{H^*} \gamma \, dH^*
\]  
(12)
Model Levels Heights designed for Geodesy

The normal gravity $\gamma$ is given on the surface of the ellipsoid at the geodetic latitude $\phi$ by the formula of Somigliana.

$$\gamma = \gamma_e \frac{1 + ksin^2\phi}{\sqrt{1 - e^2sin^2\phi}}$$

where $k = \frac{b\gamma_p}{a\gamma_e} - 1$ \hspace{1cm} (13)

The normal gravity above the ellipsoid is a function of the geodetic height $h$:

$$\gamma_h = \gamma \left(1 - 2\frac{h}{a} \left(1 + f + m - 2f \ sin^2\phi\right) + 3\frac{h^2}{a^2}\right)$$

(14)

The geodetic height $h_k(\phi, \lambda)$ of the geopotential $\Phi_k(\phi, \lambda)$ satisfy:

$$\Phi_0 - \Phi_k = \gamma h_k \left(1 - \frac{h_k}{a} \left(1 + f + m - 2f \ sin^2\phi\right) + \frac{h_k^2}{a^2}\right)$$

(15)
Refractivity of the moist air

- The refractivity of the moist air is the key physical parameter which drive the propagation of GNSS signals through the troposphere.
- The total refractivity of moist air is expressed in terms of the (total moist air) pressure and the wet (water vapor) partial pressure.

\[
N_k = k_1 \frac{p_k}{T_{vk}} + \left( k_2' \frac{p_{vap,k}}{T_k} + k_3 \frac{p_{vap,k}}{T_k^2} \right)
\]  

(16)

- This decomposition is preferred because its first term strictly follows the hydrostatic equilibrium of the model and the total refractivity does not depart two much from an hydrostatic equilibrium.
- The refractivity constants are empirically determined coefficients!
  \[
k_1 = 77.6 \times 10^{-2} K.Pa^{-1}, \quad k_2 = 70.4 \times 10^{-2} K.Pa^{-1}, \\
k_3 = 3.739 \times 10^3 K^2.Pa^{-1}, \quad k_2' = 22.1 \times 10^{-2} K.Pa^{-1} \quad (\text{Bevis et al. 1994}).
\]
Figure: Surface refractivity $N_{01}$ of the Tsukuba VLBI 7345 site 08/12/2008, 0Z.
Refractivity Functions: Vertical interpolation

Vertical refractivity functions are introduced in order to properly represent the refractivity between the model levels along the normal to the ellipsoid which points to the zenith of the local horizon.

Since the hydrostatic term follows the hydrostatic equilibrium and since the vertical (total) refractivity profile is close to an hydrostatic profile, exponential functions are chosen.

\[
N(\phi_i, \lambda_j, z) = \exp(N_k^c + N_k^z z) \quad \forall z \in [z_k, z_{k-1}] \quad (17)
\]

\[
N_k^c(\phi_i, \lambda_j) = (z_{k-1} \ln N_k - z_k \ln N_{k-1})/(z_{k-1} - z_k) \quad (18)
\]

\[
N_k^z(\phi_i, \lambda_j) = (\ln N_{k-1} - \ln N_k)/(z_{k-1} - z_k) \quad (19)
\]
Refractivity Functions: Horizontal interpolation

For a point \( P(\phi, \lambda, z) \) along the propagation path, the coefficients \( N_k^C(\phi_i, \lambda_j) \) and \( N_k^Z(\phi_i, \lambda_j) \) of the four neighbouring refractivity profiles are interpolated by a bilinear interpolation to define the vertical refractivity profile by \( N_P^c(\phi, \lambda) \) and \( N_P^z(\phi, \lambda) \) at point \( P \).

\[
N(\phi, \lambda, z) = \exp(N_P^c(\phi, \lambda) + N_P^z(\phi, \lambda)z)
\]  

(20)

The refractive index \( n(z) \) is given along the propagation path.

\[
n(\phi, \lambda, z) = 10^{-6} N(\phi, \lambda, z) + 1
\]  

(21)

The refractivity functional \( N(\phi, \lambda, z) \) provide a continuous formulation of the vertical gradient of the refractive index.

\[
\frac{dn}{dz} = (n - 1)N_P^z \text{ with } z \in [z_k, z_{k-1}]
\]  

(22)
If the tracking site is slightly below the bottom level (\(k=N_{\text{LEV}}\)) or if the propagation ends above the top level (\(k=1\)) the extrapolation is made from the continuous formulation of the nearest level.

With the purpose to estimate separately the respective impacts of the hydrostatic and non-hydrostatic terms, a similar continuous formulation is introduced only for the hydrostatic refractivity \(\bar{N}\).

\[
\bar{N}(\phi_i, \lambda_j, z) = \exp(\bar{N}_k^c + \bar{N}_k^z z) \quad \forall z \in [z_k, z_{k-1}]
\]  

(23)
Figure: Surface refractivity $N_{91}$ of the Tsukuba VLBI site (08/12/2008 0Z) and Latitudinal vertical refractivity profile from the surface to a height of 25 km computed every 125 m using vertical refractivity functions.
Conclusion: Heights & Refractivity Functions

- The ellipsoidal shape of the Earth and the Atmosphere, the geoid and gravity acceleration above the ellipsoid are taken into account.
- The model geometry is provided in ellipsoidal geodetic coordinates.
- The model levels heights are given in terms of orthometric heights.
- The model refractivities are provided by a continuous formulation by refractivity functions of the orthometric height.
2 Propagation & Ray-Tracing

- Eikonal Equation
- Ray-Tracing
- Tropospheric Delays
- Ray Bending
- Parallax
- Azimuthal Anisotropy
The eikonal equation in spherical coordinates is 

\[
H = \frac{1}{2} \left( p_r^2 + \frac{p_{\theta}^2}{r^2} + \frac{p_{\lambda}^2}{r^2 \sin^2 \theta} - n^2(r, \theta, \lambda) \right) = 0
\]  

(24)

Hamilton’s equations for the Hamiltonian are:

\[
\frac{dr}{d\sigma} = \frac{\partial H}{\partial p_r} = p_r
\]  

(25)

\[
\frac{d\theta}{d\sigma} = \frac{\partial H}{\partial p_{\theta}} = \frac{p_{\theta}}{r^2}
\]  

(26)

\[
\frac{d\lambda}{d\sigma} = \frac{\partial H}{\partial p_{\lambda}} = \frac{p_{\lambda}}{r^2 \sin^2 \theta}
\]  

(27)

\[
\frac{dp_r}{d\sigma} = -\frac{\partial H}{\partial r} = n\frac{\partial n}{\partial r} + \frac{1}{r} \left( \frac{p_{\theta}^2}{r^2} + \frac{p_{\lambda}^2}{r^2 \sin^2 \theta} \right)
\]  

(28)

\[
\frac{dp_{\theta}}{d\sigma} = -\frac{\partial H}{\partial \theta} = n\frac{\partial n}{\partial \theta} + \left( \frac{p_{\lambda}^2 \cot \theta}{r^2 \sin^2 \theta} \right)
\]  

(29)

\[
\frac{dp_{\lambda}}{d\sigma} = -\frac{\partial H}{\partial \lambda} = n\frac{\partial n}{\partial \lambda}
\]  

(30)
The independent variable $\sigma$ is the so-called generating parameter, which is related to the arclength $s$ by

$$ds = nd\sigma$$

(31)

and to the radio path length $L$ of the electromagnetic wave by

$$dT = nds = n^2 d\sigma.$$  

(32)

Introducing the elevation angle $\varepsilon$ above the local horizon and the azimuth $\alpha$ taken counterclockwise from due south at every point $P(r, \theta, \lambda)$ along the ray, the conjugate momentum variables are:

$$p_r = n \sin \varepsilon$$  

(33)

$$p_\theta = n \ r \ \cos \varepsilon \ \cos \alpha$$  

(34)

$$p_\lambda = n \ r \ \cos \varepsilon \ \sin \alpha \ \sin \theta$$  

(35)
Using the following partial derivative of $\lambda$ in term of the angles $\varepsilon$ and $\alpha$,

$$\frac{d\lambda}{d\sigma} = \frac{n \cos \varepsilon \sin \alpha}{r \sin \theta}$$  \hspace{1cm} (36)

the six first-order eikonal differential equations are further reduced, by taking the longitude $\lambda$ as the independent variable:

$$\frac{dr}{d\lambda} = \frac{r \sin \theta \tan \varepsilon}{\sin \alpha} \hspace{1cm} \frac{d\theta}{d\lambda} = \sin \theta \cot \alpha \hspace{1cm} \frac{d\lambda}{d\lambda} = 1$$  \hspace{1cm} (37)

The partial derivatives of the elevation $\varepsilon$ and the azimuth $\alpha$ are:

$$\frac{d\varepsilon}{d\lambda} = \frac{\sin \theta}{\sin \alpha} \left( \frac{1}{n} \frac{dn}{dr} + 1 \right) + \frac{\sin \theta \tan \varepsilon}{\tan \alpha} \frac{1}{n} \frac{dn}{d\theta} + \tan \varepsilon \frac{1}{n} \frac{dn}{d\lambda}$$  \hspace{1cm} (38)

$$\frac{d\alpha}{d\lambda} = -\cos \theta + \frac{\sin \theta}{\cos^2 \varepsilon} \frac{1}{n} \frac{dn}{d\theta} - \frac{\cot \alpha}{\cos^2 \varepsilon} \frac{1}{n} \frac{dn}{d\lambda}$$  \hspace{1cm} (39)
Partial derivatives of the arclength of the ray $s$ and radio path length $L$ of the wave through the dense atmosphere complete the eikonal differential system to provide the required solution.

\[
\frac{ds}{d\lambda} = \frac{r \sin \theta}{\cos \epsilon \sin \alpha} \left( = \frac{dL_s}{d\lambda} \right)
\]

\[
\frac{dL}{d\lambda} = n \frac{r \sin \theta}{\cos \epsilon \sin \alpha}
\]  

(40)  

(41)  

An additional partial derivative relative to the hydrostatic delay $\bar{D}_h$ is required to separate the hydrostatic and non-hydrostatic terms.

\[
\frac{d\bar{D}_h}{d\lambda} = \bar{n}_h \frac{r \sin \theta}{\cos \epsilon \sin \alpha}
\]

(42)  

This set of eight first-order eikonal differential equations describe the partials of the position and the direction at the current point $P(r, \theta, \lambda)$ and the partials of the lengths, solutions of the propagation problem.
Ray-Tracing: Integration of the differential system

- Raytracing is performed by the integration of this set of eight differential equations made off
  - the five differential equations describing the ray’s position in spherical coordinates,
  - the ray’s direction in the tangent local frame by the elevation $\varepsilon$ and the azimuth $\alpha$,
  - the three differential equations of the ray’s lengths.

- Integration is achieved by a fourth order Runge-Kutta scheme.
- Integration is performed in spherical coordinates when the atmospheric model is described in ellipsoidal coordinates.
Ray-Tracing: Initial Conditions

- Initial conditions are provided by the reference site position in spherical coordinates $P_{site}(r_{site}, \theta_{site}, \lambda_{site})$.
- Initial ray direction is given by the elevation $\varepsilon_{site}$ and azimuth $\alpha_{site}$.
- The elevation $\varepsilon_{spherical}$ and azimuth $\alpha_{spherical}$ are relative to the normal to the sphere and related to the elevation $\varepsilon_{ellipsoid}$ and azimuth $\alpha_{ellipsoid}$ relative to the ellipsoid normal by

$$\varepsilon_{ellipsoid} = \varepsilon_{spherical} + \left(\pi/2 - \phi - \theta\right) \cos \alpha \quad (43)$$

$$\alpha_{ellipsoid} = \alpha_{spherical} + \left(\pi/2 - \phi - \theta\right) \tan \varepsilon \quad (44)$$
During raytracing, the current point position along the ray path expressed in spherical coordinates is converted in ellipsoidal coordinates $P(\phi, \lambda, z)$ in order to retrieve the refractivity.

For a position in spherical coordinates, the height $z$ in ellipsoidal geodetic coordinates is retrieved.

The model level $k$ is determined and hence the coefficients of refractivity functions of the four neighboring columns.

The coefficients characterizing the total and hydrostatic vertical refractivity profiles $N_c^z(\phi, \lambda)$, $N_z^p(\phi, \lambda)$, $\bar{N}_c^z(\phi, \lambda)$, $\bar{N}_p^z(\phi, \lambda)$ are retrieved by bilinear interpolation.

Estimated at the height $z$, the refractive indices $n(\phi, \lambda, z)$ and $\bar{n}(\phi, \lambda, z)$ are provided to the eikonal differential system.
Ray-Tracing: Refractivity Gradients at Ray

- The vertical gradient is straightforwardly derived from the vertical refractivity profile coefficient $N^z_p$.
- The vertical gradient expressed along the normal to the ellipsoid is converted into the radial gradient along the normal to the sphere as required during the numerical integration.
- Horizontal gradients are estimated from the refractivity values at the same height $z$.
- Horizontal gradients are hundred times smaller than vertical gradient and can be neglected. The ray trajectory is inside a plane.
The arclength $L_S$ is simply the geometrical length of the ray.

$$L_S = \int_S ds$$

$$L = \int_S n\ ds = \int_S ds + \int_S (\tilde{n}_h - 1)\ ds + \int_S (\tilde{n}_{nh} - 1)\ ds$$

The radio path length $L$ account for:

- an increase of length due to the wave propagating at a speed $v = c/n$, in the medium of refractive index $n > 1$ (slower than the speed of light $c$ it would have in vacuum),

- an increase of the arclength $L_S$ of the ray path $S$, due to the bending of the ray path induced by the gradients of the refractive medium (lengthier than the straighth line path $S_L$ it would follow in vacuum).
Tropospheric Delays

The geometric delay $D_G$ is defined as the difference in length of the paths $S$ and $S_L$, the arclength $L_S$ minus the straight line $L_G$.

$$D_G = L_S - L_G = \int_S ds - \int_{S_L} ds \quad (46)$$

The chosen decomposition of the refractivity lead to the total delay $D_t$, sum of the hydrostatic $D_h$ and the non-hydrostatic $D_{nh}$ delays.

$$D_t = \int_S (n-1) \, ds \quad D_h = \int_S (\bar{n}_h-1) \, ds \quad D_{nh} = \int_S (\tilde{n}_{nh}-1) \, ds \quad (47)$$

The radio delay $D$ is the difference between the radio length $L$ and the geometric distance $L_G$.

$$D = L - L_G = \int_S n \, ds - \int_{S_L} ds \quad (48)$$
Refractive Indices & Tropospheric Delays

Figure: Variation of refractive indices & delays with altitude during ray-tracing for a specific ray $\varepsilon = 5^\circ$ and $\alpha = 90^\circ$, for Tsukuba on August 12, 2008, 00Z.
Ray Bending

Figure: Variation of Elevation and Azimuth with altitude during ray-tracing, for this specific ray, for the Tsukuba VLBI 7345 site on August 12, 2008, 00Z
Figure: The Parallax Problem
The ray-tracing is on purpose stopped on a reference sphere of radius $r_{top} = 6450$ km at the top of the troposphere. Propagation above the troposphere is here assumed to be propagation in vacuum.

To determine the ray direction which reach the satellite position $P_{sat}(r_{sat}, \theta_{sat}, \lambda_{sat}; t)$ the outgoing ray direction have to satisfy the continuity condition $\varepsilon'_{\text{ray \ top}} = \varepsilon'_{\text{out \ top}}$.

The direction of the satellite from the top sphere is provided by a family of lines which encounter the sphere with an elevation $\varepsilon'_{\text{out \ top}}$.

$$\varepsilon'_{\text{out \ top}} = \arctan \frac{r_{top} \cos \psi_{\text{top}} - r_{sat} \cos \psi_{\text{sat}}}{r_{top} \sin \psi_{\text{top}} - r_{sat} \sin \psi_{\text{sat}}}$$

The solution to this boundary condition is unique if the elevations of both family of rays are assumed to be monotonic functions.

Finding the outgoing elevation angle $\varepsilon'_{\text{top}}$ is solved iteratively.
Initial condition is provided by $\varepsilon_{\text{site}} = \varepsilon_{\text{sat}}$ (straight line problem).

The iterative process to obtain a new estimate is the following:

- AMF-D provide the top outgoing point, the center angle $\Psi_{\text{top}}$, and the outgoing direction from the satellite:

  $$
  \varepsilon'_{\text{out}} = \arctan \frac{r_{\text{top}} \cos \Psi_{\text{top}} - r_{\text{sat}} \cos \Psi_{\text{sat}}}{r_{\text{top}} \sin \Psi_{\text{top}} - r_{\text{sat}} \sin \Psi_{\text{sat}}}
  $$

- AMF-E provide the ray outgoing direction $\varepsilon'_{\text{ray}}$.

  $$
  \varepsilon'_{\text{ray}} = \varepsilon_{\text{top}} - \Psi_{\text{top}}
  $$

- A new estimate of the solution $\varepsilon_{\text{site}}$ is obtained by

  $$
  \varepsilon_{\text{new}} = \varepsilon_{\text{old}} + \varepsilon'_{\text{out}} - \varepsilon'_{\text{ray}}
  $$

- Trigonometric considerations avoid the use of a shooting method.

- The required angles to solve the parallax problem provide the outgoing ray position and direction on top of a reference sphere.
Figure: Radio Delay (left) Radio Delay with Azimuthal Mean Removed (right) for the Tsukuba VLBI 7345 site on August 12, 2008, 00Z

AMF–R–F3A4C5G2–7345–21730S007–21408.0000

AMF–R–F3A4C5G2–7345–21730S007–21408.0000
Figure: Outgoing Ray Elevation (left) with Azimuthal Mean Removed (right) for the Tsukuba VLBI 7345 site on August 12, 2008, 00Z
Conclusion: Ray-Tracing
Adaptive Mapping Functions

- Adaptive Mapping Functions
- Partial Derivatives
- Fitting Mapping Functions
- Performance of Adaptive Mapping Functions
- Mapping Elevations and Delays
- Time Discretisation
The basic underlying idea is to insert a dependency to the azimuth $\alpha$ inside each fraction coefficient $a_{i_f}$ by introducing a Fourier serie in $\alpha$:

$$a_{i_f} = a_{i_f,0} + \sum_{i_\alpha=1}^{i_\alpha=n_\alpha} a_{i_f,i_\alpha}^c \cos i_\alpha \alpha + a_{i_f,i_\alpha}^s \sin i_\alpha \alpha$$

The numerators and denominators at each level index $i_f$ are

$$N_{n_f} = 1 + a_{n_f} \quad D_{n_f} = \sin \varepsilon + a_{n_f}$$

$$N_{i_f} = 1 + a_{i_f}/N_{i_f+1} \quad D_{i_f} = \sin \varepsilon + a_{i_f}/D_{i_f+1}$$

These recursive definitions lead to the numerator $N_f$ and denominator $D_f$ of the mapping function $f$. $f$ is scaled by a scaling factor $S_f$.

$$N_f = 1 + \frac{a_1}{1 + \frac{a_2}{1 + \ldots}} \quad D_f = \sin \varepsilon + \frac{a_1}{\sin \varepsilon + \frac{a_2}{\sin \varepsilon + \ldots}} \quad f = S_f \frac{N_f}{D_f}$$

The elevation mapping functions are specific cases $f^\varepsilon = f \cos \varepsilon$. 
The usual mapping function $f_{3,0}(\varepsilon)$ truncated at the third fraction $n_f = 3$ does not depend on $\alpha (n_\alpha = 0)$,

$$f = S_f \frac{1 + \frac{a_1}{1 + \frac{a_2}{1 + a_3}}}{\sin \varepsilon + \frac{a_1}{\sin \varepsilon + \frac{a_2}{\sin \varepsilon + a_3}}} \quad (57)$$

The adaptive mapping function $f_{3,1}(\varepsilon, \alpha)$ truncated at the third fraction $n_f = 3$ with $\alpha$-series truncated at the first term $n_\alpha = 1$ is

$$N_{f_{3,1}} = 1 + \frac{a_{1,0} + a_{1,1}^c \cos \alpha + a_{1,1}^s \sin \alpha}{1 + \frac{a_{2,0} + a_{2,1}^c \cos \alpha + a_{2,1}^s \sin \alpha}{1 + a_{3,0} + a_{3,1}^c \cos \alpha + a_{3,1}^s \sin \alpha}} \quad (58)$$

$$D_{f_{3,1}} = \sin \varepsilon + \frac{a_{1,0} + a_{1,1}^c \cos \alpha + a_{1,1}^s \sin \alpha}{\sin \varepsilon + \frac{a_{2,0} + a_{2,1}^c \cos \alpha + a_{2,1}^s \sin \alpha}{\sin \varepsilon + a_{3,0} + a_{3,1}^c \cos \alpha + a_{3,1}^s \sin \alpha}} \quad (59)$$
Gradients are introduced in two alternative forms: a formulation where the gradient is embedded in the fraction form by addition of four terms in the fraction coefficient $a_1$:

$$a_1^g = a_1 + \left( E_c^c \cos \alpha + E_c^s \sin \alpha \right) \cos \varepsilon$$

$$+ \left( E_t^c \cos \alpha + E_t^s \sin \alpha \right) \tan \varepsilon$$

(60)

OR the usual formulation:

$$f = S_f \frac{N_f}{D_f} \left( 1 + \left( D_c^c \cos \alpha + D_c^s \sin \alpha \right) \cot \varepsilon \right)$$

(62)
Partial Derivatives

Partial derivative relative to each coefficient of the mapping function are needed to fit the mapping function on the traced rays. Theses partial derivatives are computed recursively and derived from the derivative of the fraction coefficient $a_{if}$.

\[
\frac{dN_1}{da_1} = \frac{1}{N_2}
\]
\[
\frac{dD_1}{da_1} = \frac{1}{D_2}
\]  
(63)

\[
\frac{dN_{if}}{da_{if}} = -\frac{a_{if-1}}{N_{if}N_{if+1}} \frac{dN_{if-1}}{da_{if-1}}
\]
\[
\frac{dD_{if}}{da_{if}} = -\frac{a_{if-1}}{D_{if}D_{if+1}} \frac{dD_{if-1}}{da_{if-1}}
\]  
(64)

\[
\frac{dN_{nf}}{da_{nf}} = -\frac{a_{nf-1}}{N_{nf}} \frac{dN_{nf-1}}{da_{if-1}}
\]
\[
\frac{dD_{nf}}{da_{nf}} = -\frac{a_{nf-1}}{D_{nf}} \frac{dD_{nf-1}}{da_{if-1}}
\]  
(65)
Partial derivatives relative to the coefficient $a^c_{i_f,0}$ which does not depend on the azimuth $\alpha$ are
\[
\frac{df}{da^c_{i_f,0}} = S_f \left( \frac{dN_{i_f}}{da^c_{i_f}} D_f - \frac{dD_{i_f}}{da^c_{i_f}} N_f \right) D_f^{-2}
\]
and in the specific case of the elevation mapping function
\[
\frac{df^\varepsilon}{da^c_{i_f,0}} = \frac{df}{da^c_{i_f,0}} \cos \varepsilon
\]
Partial derivatives relative to the $\alpha$-series coefficients are
\[
\frac{df}{da^c_{i_f,i_\alpha}} = \frac{df}{da^c_{i_f,0}} \cos i_\alpha \alpha \\
\frac{df}{da^s_{i_f,i_\alpha}} = \frac{df}{da^s_{i_f,0}} \sin i_\alpha \alpha
\]
The partial derivative relative to the scaling factor is
\[
\frac{df}{dS_f} = \frac{f}{S_f}
\]
Partial Derivatives : Gradients

The partial derivative relative to the embedded gradient is

\[
\frac{df}{dE_C^e} = \frac{df}{da_{1,0}} \cos \alpha \cos \varepsilon \\
\frac{df}{dE_C^s} = \frac{df}{da_{1,0}} \sin \alpha \cos \varepsilon \\
\frac{df}{dE_t^e} = \frac{df}{da_{1,0}} \cos \alpha \tan \varepsilon \\
\frac{df}{dE_t^s} = \frac{df}{da_{1,0}} \sin \alpha \tan \varepsilon
\] (70)

The partial derivative relative to the classical gradient is

\[
\frac{df}{dD_c} = f \cos \alpha \cot \varepsilon \\
\frac{df}{dD_s} = f \sin \alpha \cot \varepsilon
\] (72)
Determining the mapping function coefficients at a given site and a given epoch is an overdetermined and non-linear problem and require to fit iteratively all the coefficients of the mapping functions including the scale factor, the azimuthal coefficients and the gradient coefficients for the raytraced delays or the bending elevations.

The non-linear fit is achieved by the Levenberg-Marquardt method, the standard of nonlinear least-squares method.
Levenberg-Marquardt : The Merit Function

- The set of $n_r$ ray-traced delays (or bending angle) $r(\varepsilon_i, \alpha_i)$ with $i = 1...n_r$, the $\varepsilon_i$ elevation refer to the normal to the ellipsoid, is now modeled by the mapping function $f(\varepsilon_i, \alpha_i; p_1...p_{n_p})$ which nonlinearly depends on the set of $n_p$ unknown parameters $p_j$ with $j = 1...n_p$ of the function $f$.

$$\{p_1, \ldots, p_{n_p}\} = \{S_f, a_{i_f,0}, a_{i_f,i_\alpha}^c, a_{i_f,i_\alpha}^s, D^c, D^s\}, \quad (73)$$

- The merit function include a realistic statistical model of the elevation dependent variability of the delays.

$$\chi^2 = \sum_{i=1}^{i=n_r} \left[ \frac{r(\varepsilon_i, \alpha_i) - f(\varepsilon_i, \alpha_i; p_1...p_{n_p})}{{\sigma}_i} \right]^2 \quad (74)$$
The gradient and a second partial derivation of the $\chi^2$ merit function are used to take a step down (the gradient) and to define the Hessian matrix and the second member, leading to the set of linear equations:

$$H_{k,l} = \frac{1}{2} \frac{\partial^2 \chi^2}{\partial p_k \partial p_l} \quad S_k = -\frac{1}{2} \frac{\partial \chi^2}{\partial p_k} \sum_{l=1}^{l=n_p} H_{k,l} \delta p_l = S_k \forall k \in \{1...n_p\} \quad (75)$$

Some issues lead to remove second derivatives from $H$.

$$H_{k,l} = \sum_{i=1}^{i=n} \frac{1}{\sigma_i^2} \frac{\partial f(\varepsilon_i, \alpha_i; p_1...p_{n_p})}{\partial p_k} \frac{\partial f(\varepsilon_i, \alpha_i; p_1...p_{n_p})}{\partial p_l} \quad (76)$$

The idea of the Levenberg-Marquardt method is to force the modified Hessian matrix $H'$ to be diagonally dominant by multiplying the diagonal terms of $H$ by a factor $(1 + \kappa)$.

$$H'_{l,l} = H_{l,l}(1 + \kappa) \text{ and } H'_{k,l} = H_{k,l} \forall k \neq l \quad (77)$$
Levenberg-Marquardt : Iterative Scheme

- The linear system $H' \delta p = S$ is solved iteratively.
- Given an initial guess $p$ and an initial $\kappa = 0.001$, the merit function $\chi^2(p)$ is computed.
- The iterative scheme takes two steps.
  - The linear system is solved for $\delta p$ and $\chi^2(p + \delta p)$ is evaluated.
  - A convergence test is performed on the merit function:
    - If $\chi^2(p + \delta p) \geq \chi^2(p)$, $\kappa$ is increased by a factor ten, else
    - if $\chi^2(p + \delta p) < \chi^2(p)$, $\kappa$ is decreased by a factor ten
    and the trial solution is updated $p = p + \delta p$.
- Iterations continue until the merit function decrease less than $10^{-6}$. 
Performance of Adaptive Mapping Functions

- Several fraction truncations and $\alpha$-series truncations are now tested, including classical mapping functions.
- A convention is useful to name specific mapping functions:
  - F3 is the fraction truncation number $n_f = 3$,
  - A4 is the truncation of the serie at $n_\alpha = 4$,
  - C5 is the elevation cutoff at 5 degrees,
  - G2 is the two terms classical gradient and
  - G4 the embedded gradient form.
- Fitted Mapping Functions:
  - AMF-D: Site Elevation minus Outgoing Point Elevation
  - AMF-E: Site Elevation minus Outgoing Ray Elevation
  - AMF-H: Hydrostatic Delay
  - AMF-N: Non-hydrostatic Delay
  - AMF-T: Total Delay = Hydrostatic + Non-Hydrostatic Delays
  - AMF-R: Radio Delay = Total + Geometric Delays
Figure: (up) Radio AMF-R-F3A0C5G2 (left) Hydro AMF-H-F3A0C5G2 (right) (down) Total AMF-T-F3A0C5G2 (left) Non-Hydro AMF-N-F3A0C5G2 (right) for the Tsukuba VLBI 7345 site on August 12, 2008, 00Z
Figure: (up) Radio AMF-R-F3A2C5G2 (left) Hydro AMF-H-F3A2C5G2 (right) (down) Total AMF-T-F3A2C5G2 (left) Non-Hydro AMF-N-F3A2C5G2 (right) for the Tsukuba VLBI 7345 site on August 12, 2008, 00Z.
Figure: (up) Radio AMF-R-F3A4C5G2 (left) Hydro AMF-H-F3A4C5G2 (right) (down) Total AMF-T-F3A4C5G2 (left) Non-Hydro AMF-N-F3A4C5G2 (right) for the Tsukuba VLBI 7345 site on August 12, 2008, 00Z
Figure: (up) Radio AMF-R-F5A6C5G2 (left) Hydro AMF-H-F5A6C5G2 (right) (down) Total AMF-T-F5A6C5G2 (left) Non-Hydro AMF-N-F5A6C5G2 (right) for the Tsukuba VLBI 7345 site on August 12, 2008, 00Z.
Figure: AMF-D & AMF-E & AMF-R (F3A4C5G2) for the PARALLAX problem (up) mapping function fit & residuals; (down) gradient & azimuthal anisotropy; for the Tsukuba VLBI 7345 site on August 12, 2008, 00Z
Refractivity of the Troposphere
Propagation & Ray-Tracing
Adaptive Mapping Functions
Conclusions

Adaptive Mapping Functions
Partial Derivatives
Fitting Mapping Functions
Performance of Adaptive Mapping Functions
Mapping Elevations and Delays
Time Discretisation

Moyenne : -0.3285E-03
Ecart-type : 0.3788E-01
Nombres de points : 1547

Azimuthal Anisotropy Adaptive Mapping Functions
Figure: Azimuthal anisotropy of the tropospheric radio delay for Tsukuba site for 24 hourly steps beginning on August 11, 2008, 21Z until August 12, 20Z.
Conclusions

- This study provide a detailed discussion of some physical and mathematical formulations needed to maintain as far as possible a millimetric accuracy in handling the atmospheric model, ray-tracing, bending, fitting mapping functions and solving the parallax problem.

- As the tropospheric delays are very sensitive to elevation, especially at low elevation, the parallax problem has to be properly solved.

- Determining the proper ray elevation at the site is as critical as properly fitting the azimuthal anisotropy.
Conclusions

- Although the millimetric accuracy is a concern which motivates the introduction of some formulations, the time discretization may not guarantee such precision and further investigations are required.
- Undergoing investigations of a large number of meteorological situations depending on the site and the season and practical orbitographic studies should provide realistic performances of Azimuthal Anisotropy Adaptive Mapping Functions.